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2005 J. Phys. A: Math. Gen. 38 6587

(http://iopscience.iop.org/0305-4470/38/29/013)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 6587–6601 doi:10.1088/0305-4470/38/29/013

Maximal suppression of decoherence in Markovian quantum systems

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Received 3 February 2005, in final form 10 June 2005
Published 6 July 2005
Online at stacks.iop.org/JPhysA/38/6587

Abstract

In this paper, we design an optimal control law to suppress decoherence effects in Markovian open quantum systems. The optimal control law is subject to the tracking precision of the trajectory governed by the free system, which is ideally free from decoherence. We observe from numerical simulation that the undesired decohering dynamics can be partially squeezed out in most systems. Moreover, we observe the existence of sinusoidally oscillating resonant modes that play dominant roles in the controlled trajectory, which can be easily realized by continuous wave pulses. These key features are strictly demonstrated in subsequent analysis under proper assumptions. For systems in which the coherent control does not work, we suggest a feedback control strategy to extend the applicability of control to wider class of systems.

PACS numbers: 03.67.Lx, 03.67.Pp

1. Introduction

In recent years, rapidly increasing research on quantum information technologies [1] has been dedicated to theoretical and experimental applications to microscopic systems [2–7]. Among various unsolved problems, decoherence is recognized as a bottleneck for its severe destruction of quantum superposition that is the key to quantum information processing. Various schemes have been proposed to reduce this unexpected effect. The quantum error-correction code [8–12] and error-avoiding code [13–15] schemes are presented by encoding quantum states onto the carefully selected subspaces that are isolated from the decoherence channels. Such ideas evolve into the continuous error correction via feedback of the classical information extracted from continuous quantum measurement [16, 17]. In addition, Viola and Lloyd proposed the fast switching bang–bang controls [18–21] borrowing ideas from refocusing techniques.

It is convenient to employ optimal control technique in decoherence control, which has been successfully applied to laser control of bond-selection chemical systems [22, 23]. Along this line, several studies have been done to seek control solutions to reduce the decoherence effects by minimizing a certain class of cost functionals [24–26]. In this paper, we shall adopt the method of optimal trajectory tracking in order to force the system state to evolve as 'close' as possible to the expected quantum process. The paper is organized as follows: in section 2, we formulate the control problem in the representation of real coherent vector. In section 3, we apply optimal tracking method to numerically simulate three fundamental decoherence models in quantum computation. In section 4, we introduce the concept of stationary solution and study the special case of sinusoidal stationary control laws that suffice to be a good approximation to the optimal control laws. In section 5, feedback control strategy is introduced to treat exceptional cases. The conclusion is presented in section 6.

2. Bloch vector model

In this paper, we consider the Markovian open quantum systems that are widely studied such as in [27–29]. The mathematical model can be written as the following master equation [30]:

$$\dot{\rho} = -\mathrm{i}[H_0, \rho] - \mathrm{i}\sum_{i=1}^{n} u_i[H_i, \rho] + \sum_{j=1}^{m} \Gamma_j D[L_j]\rho, \tag{1}$$

where the Planck constant $\hbar$ has been assigned to be 1. The quantum state is represented by the density matrix ρ. H_0 refers to the free Hamiltonian and $H_1, \ldots, H_n$ are the control Hamiltonians adjusted by control parameters $u_1, \ldots, u_n$, respectively. The Lindblad terms $D[L_j]\rho = L_j\rho L_j^\dagger - \frac{1}{2}L_j^\dagger L_j\rho - \frac{1}{2}\rho L_j^\dagger L_j$, where $j = 1, \ldots, m$, characterize the dissipative channels via interactions of system operators L_j with the environment. The positive coefficients Γ_j represent the damping rates.

Note that the Markovian approximation adopted in (1) is not always satisfied such as in [31–34], especially in solid-state systems [35]. The analysis of such non-Markovian systems require the information of the evolution of an external reservoir, which in general has infinite number of degrees of freedom. Hence, the corresponding control design becomes much more complicated. Nevertheless, in some circumstances [31, 36], one can make use of the 'system + bath + environment' model, in which the systems interact directly with a low-dimensional intermediate bath, and the bath interacts weakly with the external environment. With this model, the 'system + bath' part ('dressed' system) can be approximated to be Markovian, while evolution of the system itself is non-Markovian. Some other possible approaches can be seen with the help of the non-Markovian master equation [37].

Further, we assume that the quantum system in the absence of decoherence, namely

$$\dot{\rho} = -\mathrm{i}[H_0, \rho] - \mathrm{i}\sum_{i=1}^{n} u_i[H_i, \rho], \tag{2}$$

is strongly controllable [30, 38], i.e. the system (2) can be steered starting from an arbitrary pure state to any pure state at any positive time. In other words, one can drive the system arbitrarily fast along trajectories governed by any prescribed Hamiltonian. This assumption is particularly essential in the coherent control of open systems because the ability of guiding coherent dynamics is required to be as strong as possible to resist decoherence. However, it should be noted that the assumption of strong controllability imposed on the closed system (2) is not applicable to the open system (1). In fact, the strong controllability of the open system (1), i.e. controlling the system as quickly as possible, relates to the small-time behaviour of the system which may contradict the Markovian assumption describing the long-time behaviour of the open system. This fact has been rigorously demonstrated by Altafini [30] by showing that the open system is never strongly controllable.

In order to facilitate the calculations, we will convert the differential equation (1) from the complex density matrix representation into the so-called coherent vector representation [28, 30]. Firstly, we choose an orthonormal basis of $N \times N$ matrices $\{I, \Omega_j\}_{j=1,\ldots,N^2-1}$ with respect to the inner product $\langle X, Y\rangle = \mathrm{tr}(X^\dagger Y)$, where I is the N-dimensional identity matrix and Ω_j are $N \times N$ Hermitian traceless matrices. With this basis, any $N \times N$ complex matrix A can be expanded as

$$A = a_0 I + \sum_i a_i \cdot \Omega_i.$$

In particular, the Hermitian density matrix ρ can be represented as $\rho = \frac{1}{N} I + \sum_i m_i \cdot \Omega_i$. $\vec{m} = (m_1, \ldots, m_{N^2-1})^T$ is a real $(N^2 - 1)$-dimensional vector which is called the coherent vector corresponding to ρ. The system equation (1) can then be rewritten as a differential equation of the coherent vector:

$$\dot{m}(t) = O_0 m(t) + \sum_{i=1}^{n} u_i O_i m(t) - D m(t) + g, \tag{3}$$

where $O_0, O_i \in so(N^2 - 1)$ are converted from the Hamiltonian parts [30] in (1). The term '$-Dm + g$' comes from the decohering process represented by the Lindblad terms. For simplicity, one can always decompose the matrix D as $D = D_1 + D_2$, where D_1 and D_2 are, respectively, symmetrical and antisymmetrical matrices. The symmetrical part satisfies $D_1 \geqslant 0$. Obviously, the antisymmetrical part D_2 that represents the unitary part of the Lindblad terms can be absorbed into the Hamiltonian part of equation (1) under the assumption of strong controllability of (2). One can rewrite O_0 as $O_0 + D_2$, or alternatively remove D_2 directly from (3) because it can be dynamically cancelled by the strong control operations $O_1, \ldots, O_n$. Hence, without loss of generality, we can always assume that D is symmetrical and non-negative definite.

The above approach is a generalization of the well-known Bloch representation for two-level systems. Physically, the norm of the coherent vector represents the amount of coherence in the quantum state. For a pure state, the length of corresponding coherent vector is unity and it should be shorter for a mixed state. The coherent control drives the quantum state along a sphere on which coherence is conserved, while the decohering operators pull the vector towards the origin. The inhomogeneous term g is related to the stable state of the system, e.g. the ground state in spontaneous emission.

For the Markovian open quantum system (1), we assume in addition that the system satisfies the so-called (O) conditions:

$$\text{(i) } [O_0, D] = 0, \qquad \text{(ii) } O_0 g = 0.$$

The (O) conditions have important physical implications in that the equilibrium distribution of the density matrix in the absence of controls $\rho_\infty = \frac{1}{N} I + \sum_i (m_\infty)_i \cdot \Omega_i$ commutes with the internal system Hamiltonian H_0 (see appendix A for proof). That is to say, in the energy representation, the off-diagonal entries of the system density matrix, which embody the coherence in the system, decay to zero as a result of decoherence.

Under the (O) conditions, we will discuss the selection of control Hamiltonians. It is not difficult to prove that there exists a matrix basis $\{I, \Omega_i\}$ under which the matrices O_0 and D are simultaneously block-diagonal, i.e.

$$
\begin{aligned}
O_0 &= \mathrm{diag}\left(\begin{pmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -\omega_r \\ \omega_r & 0 \end{pmatrix}, 0, \ldots, 0\right), \\
D &= \mathrm{diag}\left(\begin{pmatrix} d_1 & \\ & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} d_r & \\ & d_r \end{pmatrix}, d_{2r+1}, \ldots, d_{N^2-1}\right).
\end{aligned} \tag{4}
$$

In the following sections, we always suppose that the matrix basis is chosen to satisfy that O_0 and D are already in the above form. Since the closed system (2) is strongly controllable as assumed above in this section, it suffices to study the representing case in which the control Hamiltonians are $H_i = \frac{1}{2}\Omega_i$, $i = 1, \ldots, N^2 - 1$, without much loss of generality. This is the standard model that will be studied for decoherence suppression of open Markovian systems in the next sections of this paper.

3. Optimal trajectory tracking

In principle, the decoherence is not able to be completely removed via coherent control in Markovian open quantum system. This is because, as argued by Altafini [30], the irreversible decohering dynamics is uncontrollable under coherent control. However, it is worth estimating to what extent decoherence can be suppressed. For this purpose, we are going to explore this problem by an optimal control technique to force the system to evolve along some prescribed cohering trajectory. In this paper, we choose the typical target trajectory as the free evolution $m^0(t) = \mathrm{e}^{O_0(t-t_0)} m_0$ of the free system:

$$\dot{\rho}(t) = -\mathrm{i}[H_0, \rho(t)].$$

We then arrive at the minimization problem of the following functional:

$$J[u(t)] = \frac{1}{2}\int_{t_0}^{t_f} [|m(t) - m^0(t)|^2 + \epsilon^{-1} u^T(t) u(t)]\, \mathrm{d}t, \tag{5}$$

where $m(t)$ is subjected to equation (3). The norm $|m - m^0| = [(m - m^0)^T (m - m^0)]^{1/2}$ measures the deviation of quantum state $m(t)$ from the target state $m^0(t)$. The term $\epsilon^{-1} u^T u$ is added to prevent the control intensities from going too large, where the coefficient $\epsilon \in \mathcal{R}$ is used to achieve a balance between the tracking precision and the control constraints.

Traditionally, the above optimal control problem can be solved by the maximum principle [39], by which the optimal solution can be solved by the following differential equation with two-sided boundary values:

$$\dot{m} = O_0 m + \sum_{i=1}^{m} u_i O_i m - Dm + g, \qquad m(t_0) = m_0, \tag{6}$$

$$\dot{\lambda} = O_0 \lambda + \sum_{i=1}^{m} u_i O_i \lambda + D\lambda - m + m^0, \qquad \lambda(t_f) = 0, \tag{7}$$

$$u_i = -\epsilon \lambda^T O_i m, \qquad i = 1, \ldots, n. \tag{8}$$

In general, no analytic solution exists for this boundary value problem (BVP). Nevertheless, one can always obtain a numerical solution. To illustrate this method and give more analysis, we test several simple decoherence models that are widely used in quantum

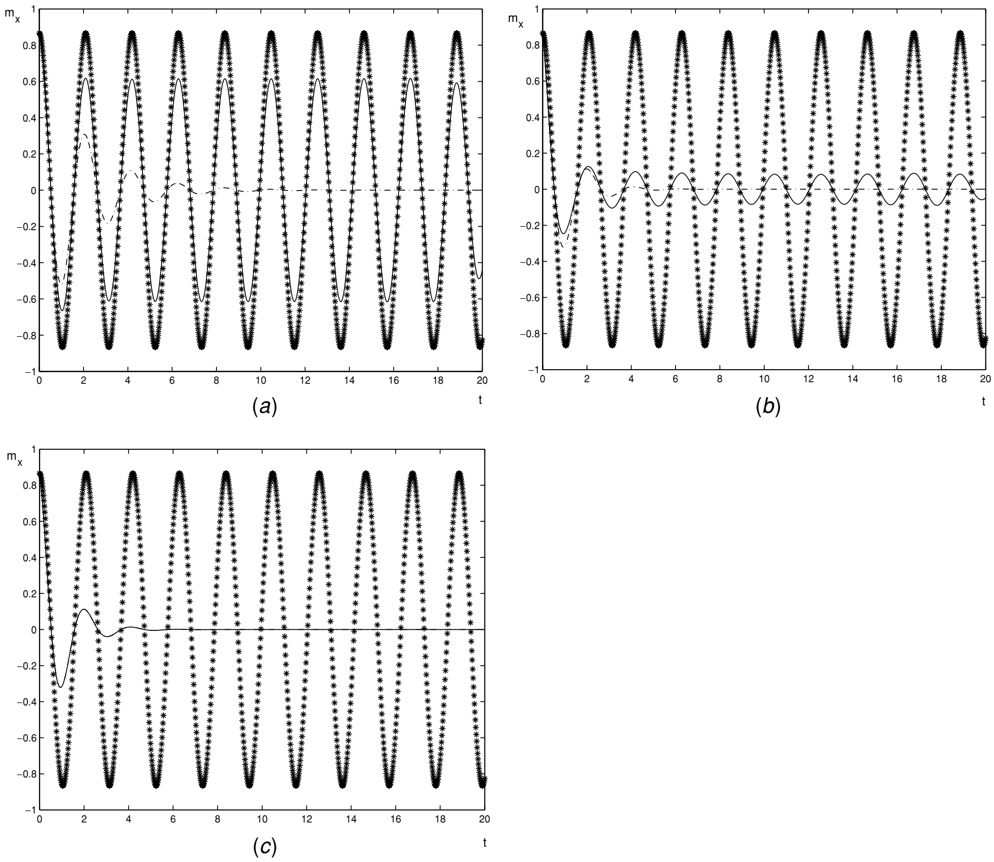

Figure 1. Temporal evolution of $m_x(t)$ for (a) amplitude damping, (b) phase damping and (c) depolarizing decohering processes. The asterisk line denotes the target trajectory $m^0(t)$, the dashed line is the uncontrolled trajectory; and the solid line is the optimally controlled trajectory.

computation theory:

(a) amplitude damping decoherence: $L = \sigma_-$, $\Gamma = 1$;
(b) phase damping decoherence: $L_1 = \frac{1}{2}(I + \sigma_z)$, $L_2 = \frac{1}{2}(I - \sigma_z)$ and $\Gamma_1 = \Gamma_2 = 1$;
(c) depolarizing decoherence: $L_1 = \sigma_x$, $L_2 = \sigma_y$, $L_3 = \sigma_z$ and $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$.

where σ_i are Pauli operators and $\sigma_- = \sigma_x - \mathrm{i}\sigma_y$. In addition, we assume that the system possesses the Hamiltonian $H = \omega\sigma_z + u_x\sigma_x + u_y\sigma_y$, where $\omega = 3$ is the Rabi frequency. The initial state is set to be a pure state $m_0 = (\sqrt{3}/2, 0, -1/2)^T$. The parameter ε is set to 1.

The simulation results are shown in figures 1 and 2. It is observed that the decohering dynamics is partially suppressed in amplitude damping and phase damping decoherence cases. For these two cases, the controlled trajectories oscillate synchronous with the target trajectories, with more or less decreasing amplitudes due to the uncontrollability of open Markovian systems. More interesting is that the resulting optimal controls also appear to be approximately sinusoidal functions as shown in figure 2(a). This feature is manifested in the power spectrum of the control intensities in figure 2(b) where a sharp peak emerges at the system resonance frequency $\omega = 3$. On the other hand, the exceptional case is the depolarizing decoherence for which the optimal control does not work (no difference can be

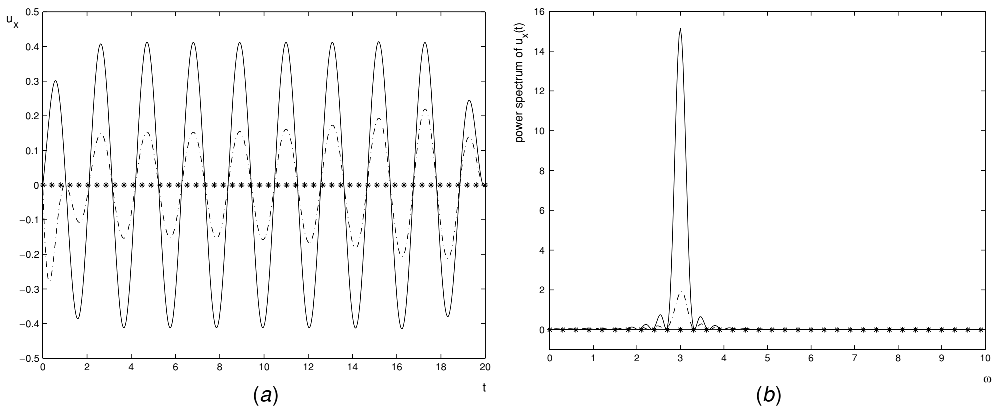

Figure 2. (*a*) The *x*-axis component of optimally designed control $u_x(t)$ and (*b*) the power spectrum of $u_x(t)$: the solid line is for the amplitude damping, the dashed line is for the phase damping and the asterisk line is for the depolarizing decohering processes.

observed between the controlled trajectory and the uncontrolled decohering trajectory in figure 1(*c*), so that they coincide). These features provide very useful hints and will be made into rigorous demonstrations in the following sections.

4. Stationary control law

The most important fact we observe from the simulation results is that the optimal control laws, as well as the controlled trajectories, are approximately sinusoidal functions of time. Recall that this property has been demonstrated in [40] for closed two-level quantum systems; we will show that open Markovian systems also have similar properties. To proceed with the subsequent analysis, we need to introduce a new concept.

Definition 1. *Consider the following nonlinear equations by substituting (8) into (6) and (7):*

$$\begin{cases} \dot{m} = -Dm + O_0 m + \sum_{i=1}^{n} \epsilon (m^T O_i \lambda) O_i m + g, \\ \dot{\lambda} = O_0 \lambda + D\lambda - m + m^0 + \sum_{i=1}^{n} \epsilon (m^T O_i \lambda) O_i \lambda. \end{cases} \tag{9}$$

A function of time $m_{ss}(t)$ starting from some initial value $m_{ss}(t_0)$ is called an exponential stationary solution of (9), if it fulfils (9) and there exist positive constants M and β such that the deviation from the real system trajectory is bounded as follows:

$$|m(t) - m_{ss}(t)| \leqslant M\, e^{-\beta(t-t_0)} |m(t_0) - m_{ss}(t_0)|.$$

The following theorem provides a sufficient condition for the existence of an exponential stationary solution (see appendix B for the rigorous proof).

Theorem 1. *There exists a positive number M such that $|m(t)|, |\lambda(t)| < M$. Denote $L = 1 + \epsilon n M^2$ and $\beta = \min\{d_1, \ldots, d_{N^2-1}\}$ being the minimum, where $\{d_i\}$ are eigenvalues of matrix D. If $\beta > 4L$, for any solutions $m^1(t), \lambda^1(t)$ and $m^2(t), \lambda^2(t)$ with boundary values*

$m^1(t_0), \lambda^1(t_f)$ and $m^2(t_0), \lambda^2(t_f)$, we have

$$|m^1(t) - m^2(t)| \leqslant \frac{\beta}{\beta - 2L} \mathrm{e}^{-\frac{\beta}{2}(t-t_0)} |m^1(t_0) - m^2(t_0)|,$$

$$|m^1(t_f - t) - m^2(t_f - t)| \leqslant \frac{\beta}{\beta - 2L} \mathrm{e}^{-\frac{\beta}{2}(t_f - t)} |m^1(t_f) - m^2(t_f)|.$$

Remark 1. The conditions in theorem 1 require that, to guarantee the convergence of the controlled trajectory, the eigenvalues of D should be no less than a lower bound $4L$. From the definition of L, this in turn sets an upper bound for ϵ, i.e. $\epsilon < (nM^2)^{-1}(\beta/4 - 1)$. Obviously, this restricts the tracking precision adjusted by ϵ in the cost functional (5). Hence, the condition is rather stringent for the existence of a stationary solution. Nevertheless, we believe that this condition can be relaxed and, as will be seen in the next section, the system can be transformed to meet this condition via feedback controls.

Theorem 1 assures the convergence of the controlled trajectory, based on which it is possible to obtain an explicit controlled stationary trajectory, i.e. the stationary solution of (9). One notes that there exists a countable number of oscillation modes for stationary solutions. Although each set of different boundary values gives a different stationary solution and the collection of all stationary solutions may be uncountable, the stationary solution can be expanded as a Fourier series with frequencies as a integral linear combination of frequencies of $m^0(t)$ in (9) and $m^0(t)$ consists of a finite number of oscillation modes. For this reason, it is seen that the oscillation modes of the stationary solution are countable. However, under the (O) conditions introduced in section 2, we can seek stationary solutions in a simple form that contains only oscillation modes that resonate with the target trajectory $m^0(t)$.

Theorem 2. *Suppose the Markovian open quantum system (1) satisfies (O) conditions: $[O_0, D] = 0$ and $O_0 g = 0$. Under the assumptions in theorem 1, there exists a stationary solution $m_{ss}(t) = \mathrm{e}^{O_0(t-t_0)}\xi$ and $\lambda_{ss}(t) = \mathrm{e}^{O_0(t-t_0)}\eta$ of (9), where $\xi, \eta \in \mathcal{R}^{N^2-1}$ satisfy*

$$-D\xi + \epsilon \sum_{i=1}^{N^2-1} (\xi^T O_i \eta) O_i \xi + g = 0, \qquad D\eta - \xi + \epsilon \sum_{i=1}^{N^2-1} (\xi^T O_i \eta) O_i \eta + m_0 = 0, \tag{10}$$

and the resulting controls can be expressed as

$$u_{2k-1}^{ss}(t) = \epsilon \xi^T O_{2k-1} \eta \cos \omega_k (t - t_0) - \epsilon \xi^T O_{2k} \eta \sin \omega_k (t - t_0),$$
$$u_{2k}^{ss}(t) = \epsilon \xi^T O_{2k-1} \eta \sin \omega_k (t - t_0) + \epsilon \xi^T O_{2k} \eta \cos \omega_k (t - t_0),$$
$$u_l^{ss}(t) = \epsilon \xi^T O_l \eta, \qquad k = 1, \ldots, r, \quad l = 2r + 1, \ldots, N^2 - 1,$$

where $\pm \mathrm{i}\omega_k$ are non-zero eigenvalues of O_0.

The proof can be found in appendix C. Obviously, the original optimization problem is reduced to a much simpler algebraic equation that results in a sub-optimal control. The theorem also conforms to our aforementioned observation, i.e. under proper assumptions and boundary values, the controlled trajectory contains typical oscillating modes resonating with the target trajectory, and the oscillating amplitudes do not decrease as time goes by. This special trajectory can be taken as an approximated optimally controlled trajectory, since the real system does not necessarily have the same boundary conditions.

Remark 2. As a trivial case, one can directly obtain the solution of the depolarizing decoherence model in which $g = 0$. It is easy to see that $\xi = 0$ by solving (10). This leads to the vanishing of the optimal controls, i.e. $u_i^{ss}(t) = 0, i = 1, 2, \ldots, N^2 - 1$. This

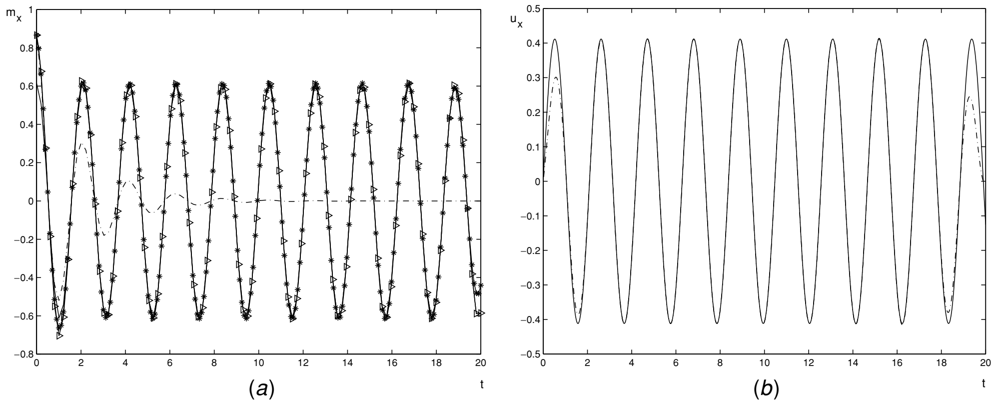

Figure 3. The figure of the amplitude damping decoherence process. (*a*) Curves of $m_x(t)$: the asterisk line denotes the optimally controlled trajectory in the example at the end of section 3, the dashed line is the decohering trajectory without control, the solid line is the stationary trajectory and the triangle line is the decohering trajectory with approximate control. (*b*) The x-axis component of controls, $u_x(t)$: the dashed line denotes the numerical control law in the example at the end of section 3 and the solid line is the stationary control law.

conforms to the simulation results in figures 1(*c*) and 2(*a*), i.e. no coherent control can resist the depolarizing decohering effects. In the next section, we will present another method to break through this limitation.

Next, we demonstrate the applicability of theorem 2 with the two-level decoherence models discussed in section 3. It is easy to verify that these systems all satisfy (O) conditions. Consider the same amplitude damping decoherence model in section 3. It can be calculated that $m_x^{ss} = 0.6150\cos\omega t$, $m_y^{ss} = 0.6150\sin\omega t$, $m_z^{ss} = -0.7468$ and $u_x^{ss} = 0.4117\sin\omega t$, $u_y^{ss} = -0.4117\cos\omega t$, where the coefficients are solved numerically from equations (10). From the simulation results shown in figure 3, we see that the stationary solution coincides with the optimally controlled trajectory. Moreover, we show the trajectory driven by the approximated controls $u_x^{ss}(t)$ and $u_y^{ss}(t)$. As a sub-optimal solution, the approximated controls perform pretty well that the resulting state trajectory (triangle line) is so close to the optimally controlled trajectory (the asterisk line) that they coincide in figure 3. This simple example verifies our conclusion.

5. Feedback control modified strategy for the optimal control

As mentioned in remarks 1 and 2, we need to separately deal with the two exceptional cases: (1) the eigenvalues of D are very small and (2) the inhomogeneous term g disappears in equation (3). In the first case, we are not able to make sure whether or not the coherent controls still work since the sufficient condition in theorem 1 does not hold to guarantee the existence of stationary solutions. In the second case, without the inhomogeneous term g, the remaining part $-Dm$ corresponding to the Lindblad terms will pull the coherent vector irreversibly towards the origin of $\mathcal{R}^{N^2-1}$, and this radial shrinking is not able to be compensated by coherent control because the corresponding orthogonal matrices $O_i \in so(N^2-1)$ can produce only angular motions that are perpendicular to that of D. To break through this obstacle, additional non-unitary dynamics needs to be introduced into the open quantum control system to alter the decohering dynamics and then make the coherent control more effective. This can be achieved by a feedback control strategy that involves quantum measurements. Suppose an observable A is continuously measured and a unitary transformation U is imposed on the system once a quantum jump is detected. In the limit of Markovian feedback, the master equation (1) can be modified as [41]

$$\dot{\rho} = -\mathrm{i}\left[H_0 + \sum_{i=1}^{n} u_i H_i, \rho\right] + \sum_j \Gamma_j D[L_j]\rho + \Gamma_A D[UA]\rho, \tag{11}$$

where the coefficient Γ_A reflects the measurement and feedback strength and the feedback efficiency. The back-action of quantum measurement and subsequent feedback action can be read in the extra term $\Gamma_A D[UA]\rho = \Gamma_A\big(UA\rho AU^\dagger - \frac{1}{2}A^2\rho - \frac{1}{2}\rho A^2\big)$, where the first term represents the detected quantum jump after which a feedback transformation is performed and the latter two terms represent the measured system dynamics without the occurrence of a quantum jump, and hence no act will take place. Note that the Markovian assumption describing the long-time behaviour, not for the small-time case, of the feedback control system (11) does not contradict the strongly controllable assumption introduced in section 2, because the strongly controllable assumption is not imposed on the open system (11), which relates to the small-time behaviour of the open systems, but on the closed system (2).

Since the measurement and feedback operation can be chosen in advance, much more freedoms are available in controlling the quantum system. Roughly speaking, one can choose proper measured observable and feedback action so that a non-zero, inhomogeneous vector g can be generated when we convert the matrix equation (11) into the corresponding coherent vector equation. In this case, the system can be affected by coherent control. This is actually equivalent to shifting and stabilizing the system to a new equilibrium state [42]. Moreover, the eigenvalues of D corresponding to (11) can be assigned to be arbitrary prescribed values, i.e. the convergence rate to the stationary can be adjusted by elaborate designs. Following these ideas, we can firstly find the required $N \times N$ complex matrix $L = UA$ that produces such g and D terms. Next, we can apply the polar decomposition [1] of L, i.e. decomposition of L into the product of a Hermitian matrix A and a unitary matrix U, by which the measured observable A and feedback action U can be directly determined.

For example, we can choose $L = \sigma_-$ for two-level depolarizing decohering systems, by which a non-zero inhomogeneous vector g_L is produced. The corresponding polar decomposition $L = \sigma_x\big[\frac{1}{2}(-I + \sigma_z)\big]$ gives the measurement observable $A = \frac{1}{2}(-I + \sigma_z)$ and feedback action $U = \sigma_x$. Under this design of feedback control, simulation results are given in figure 4. In this demonstration example, the coherent dynamics can be partially recovered. However, the improvement of decoherence suppression by feedback control is still limited. Further numerical studies show that the maximal decoherence suppression via the above feedback strategy is achieved at $\Gamma_A = 1.146$. Even in this case, the effect is not remarkably improved. Physically, there is a trade-off for the choice of the coefficient Γ_A. The larger Γ_A is, the closer we can shift the system to a pure equilibrium state, but in the meantime, the more severely the coherence is destroyed by induced non-unitary dynamics.

6. Discussion

In summary, we analyse the resonance phenomena appeared in the optimal control of decohering dynamics. From these results, we find relatively simple approaches to designing optimal controls that track prescribed trajectories for the Markovian systems. Although this method is approximate, it can be taken as a starting point of decoherence control from which the tracking error can be further corrected by other techniques such as perturbation theory and learning algorithms. We also introduce the feedback control to circumvent some exceptional

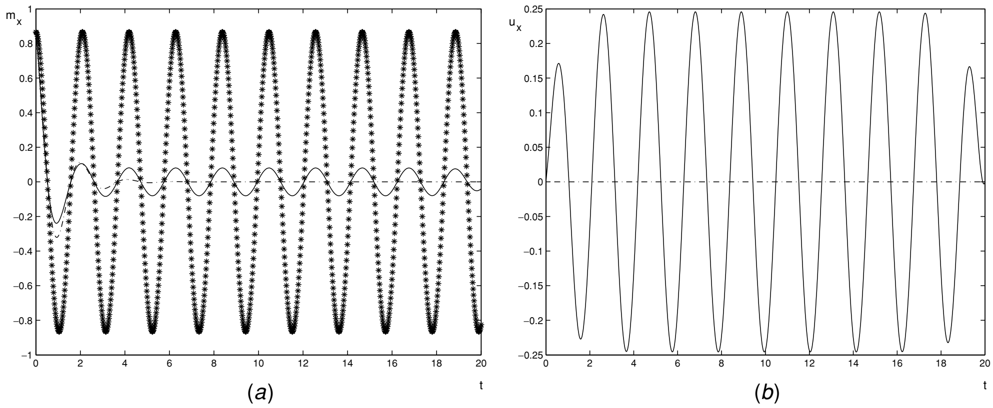

Figure 4. Feedback control of depolarizing decoherence: $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1, \Gamma_A = 1, t_0 = 0$, $t_f = 20, \omega = 3, m_0 = (\sqrt{3}/2, 0, -1/2)^T$. (*a*) Curves of $m_x(t)$: the asterisk line denotes the target trajectory $m^0(t)$; the dashed line is the numerical controlled trajectory in the example at the end of section 3 and the solid line is the trajectory with feedback and optimally designed control imposed. (*b*) The x-axis component of controls, $u_x(t)$: the dashed line denotes the numerical control law in the example at the end of section 3 and the solid line is the optimal control law with feedback.

cases such as the depolarizing decoherence. It should be noted here that the quantum feedback control alters the essential structure of the system so that the ability of coherent control can be enhanced.

In this paper, we restrict the discussion to Markovian systems and show the validity of our control strategy. In the future, it is worth extending the scope to more prevalent non-Markovian systems. The modelling of non-Markovian systems has been widely studied in the literature such as [37]. Comparing with the models we study in this paper, the matrix D and the vector g become time-variant, and the algebraic equation for the stationary solution turns into an integro-differential equation. However, by a dress-state approach this problem will still be tractable and we believe that the resonance phenomena should show themselves in such a system. Moreover, the system free Hamiltonian and control Hamitonians in equation (1) are time independent. The generalization to systems with time-dependent Hamiltonians is an interesting topic and very important in experimental studies [27], and the time dependence is essential to induce multi-photon absorption process. Further considerations of time-dependent quantum systems can be seen in [20, 43–45] including controllability studies. We will explore these problems in our future works.

Acknowledgments

This research was supported in part by the National Natural Science Foundation of China under grant numbers 60433050 and 60274025. TJT would also like to acknowledge partial support from the US Army Research Office under grant W911NF-04-1-0386.

Appendix A. Proof of the physical implication of (O) conditions

To prove the fact $[H_0, \rho_\infty] = 0$, we first observe that the uncontrolled stationary distribution ρ_∞ satisfies

$$\dot{\rho}_\infty = -\mathrm{i}[H_0, \rho_\infty] + \sum_{j=1}^{m} \Gamma_j D[L_j]\rho_\infty = 0,$$

which is equivalent to $(-D + O_0)m_\infty + g = 0$ in the coherent vector representation. From the (O) conditions $[O_0, D] = 0$ and $O_0 g = 0$, it can be deduced that $(-D + O_0)O_0 m_\infty = 0$. From equation (4), we can assume that

$$O_0 = \begin{pmatrix} O_0^1 & \\ & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix},$$

where

$$O_0^1 = \operatorname{diag}\left(\begin{pmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -\omega_r \\ \omega_r & 0 \end{pmatrix}\right),$$

and D_1 and D_2 are, respectively, $2r \times 2r$ and $(N^2 - 1 - 2r) \times (N^2 - 1 - 2r)$ non-negative diagonal matrices. Assuming $m_\infty = \left(\left(m_\infty^1\right)^T, \left(m_\infty^2\right)^T\right)^T$, where $m_\infty^1 \in \mathcal{R}^{2r}$ and $m_\infty^2 \in \mathcal{R}^{N^2-1-2r}$, we have $\left(-D_1 + O_0^1\right)O_0^1 m_\infty^1 = 0$. Since the matrix $-D_1 + O_0^1$ is reversible, it is known that $O_0^1 m_\infty^1 = 0$, i.e. $O_0 m_\infty = 0$, which means $[H_0, \rho_\infty] = \sum_i (O_0 m_\infty)_i \cdot \Omega_i = 0$.

Appendix B. Proof of theorem 1

We use the iterative algorithm to approximately solve (9), which results in the following sequence of equations:

$$m^{k,(1)}(t) = \mathrm{e}^{-D(t-t_0)} m^k(t_0) + \int_{t_0}^{t} \mathrm{e}^{-D(t-t_1)} g \, \mathrm{d}t_1,$$

$$\lambda^{k,(1)}(t) = \mathrm{e}^{-D(t_\mathrm{f}-t)} \lambda^k(t_\mathrm{f}) + \int_{t}^{t_\mathrm{f}} \mathrm{e}^{-D(t_1-t)} m^0(t_1) \, \mathrm{d}t_1,$$

$$m^{k,(m)}(t) = \mathrm{e}^{-D(t-t_0)} m^k(t_0) + \int_{t_0}^{t} \mathrm{e}^{-D(t-t_1)} g \, \mathrm{d}t_1 + \int_{t_0}^{t} \mathrm{e}^{-D(t-t_1)} \times \left[O_0 m^{k,(m-1)}(t_1) + \sum_{i=1}^{n} \epsilon((m^{k,(m-1)})^T O_i \lambda^{k,(m-1)}) O_i m^{k,(m-1)} \right] \mathrm{d}t_1,$$

$$\lambda^{k,(m)}(t) = \mathrm{e}^{-D(t_\mathrm{f}-t)} \lambda^k(t_\mathrm{f}) + \int_{t}^{t_\mathrm{f}} \mathrm{e}^{-D(t_1-t)} m^0(t_1) \, \mathrm{d}t_1 + \int_{t}^{t_\mathrm{f}} \mathrm{e}^{-D(t_1-t)} \times \left[O_0 \lambda^{k,(m-1)} - m^{k,(m-1)} + \sum_{i=1}^{n} \epsilon((m^{k,(m-1)})^T O_i \lambda^{k,(m-1)}) O_i \lambda^{k,(m-1)} \right] \mathrm{d}t_1,$$

where $k = 1, 2$ and $m = 1, 2, \ldots, \infty$. $\{m^{k,(m)}(t)\}, k = 1, 2$, are Cauchy sequences and $m^{k,(m)}(t) \to m^k(t)$, when $m \to \infty$.

We first prove that there exists $M > 0$ such that $|m^k(t)|, |\lambda^k(t)| \leqslant M$. For any initial-state $m^k(t_0)$, it is known that

$$\rho^k(t_0) = \frac{1}{N} I + \sum_i (m^k(t_0))_i \cdot \Omega_i \Rightarrow \operatorname{tr}(\rho^k(t_0))^2$$

$$= \frac{1}{N^2} + |m^k(t_0)|^2 \leqslant 1 \Rightarrow |m^k(t_0)| \leqslant 1,$$

which means there exists $M_1 > 0$ such that $|m^{k,(1)}(t)| \leqslant \mathrm{e}^{-\frac{\beta}{2}(t-t_0)} M_1$. For the same reason, there exists $M_2 > 0$ such that $|\lambda^{k,(1)}(t)| \leqslant \mathrm{e}^{-\frac{\beta}{2}(t_\mathrm{f}-t)} M_2$ for arbitrary final state $\lambda^k(t_\mathrm{f})$. Let

$M = 2\max\{M_1, M_2\}$, $L = 1 + \epsilon n M^2$. Note that for $\frac{2L}{\beta} < \frac{1}{2}$, it can be inductively proved that

$$|m^{k,(m)}(t)| \leqslant \mathrm{e}^{-\frac{\beta}{2}(t-t_0)}\left[1 + \cdots + \left(\frac{1}{2}\right)^{m-1}\right] M_1 \leqslant M.$$

Let $m \to \infty$, we have $|m^k(t)| \leqslant M$ and $|\lambda^k(t)| \leqslant M$.

Next, it can also be proved by induction that

$$|m^{1,(m)}(t) - m^{2,(m)}(t)| \leqslant (1 + \cdots + \mu^{m-1})\,\mathrm{e}^{-\frac{\beta}{2}(t-t_0)}|m^1(t_0) - m^2(t_0)|,$$

where $\mu = \frac{2L}{\beta} < \frac{1}{2}$. Since $\mu = \frac{2L}{\beta} < \frac{1}{2}$, let $m \to \infty$, it can be verified that

$$\begin{aligned}|m^1(t) - m^2(t)| &\leqslant \frac{1}{1-\mu}\,\mathrm{e}^{-\frac{\beta}{2}(t-t_0)}|m^1(t_0) - m^2(t_0)| \\ &= \frac{\beta}{\beta - 2L}\,\mathrm{e}^{-\frac{\beta}{2}(t-t_0)}|m^1(t_0) - m^2(t_0)|.\end{aligned}$$

The proof of the inequality $|m^1(t_\mathrm{f} - t) - m^2(t_\mathrm{f} - t)| \leqslant \frac{\beta}{\beta-2L}\,\mathrm{e}^{-\frac{\beta}{2}(t_\mathrm{f}-t)}|m^1(t_\mathrm{f}) - m^2(t_\mathrm{f})|$ is similar from the fact

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m^k(t_\mathrm{f} - t) &= Dm^k(t_\mathrm{f} - t) - O_0 m^k(t_\mathrm{f} - t) \\ &\quad - \sum_{i=1}^{l}\epsilon(m^k(t_\mathrm{f} - t)^T O_i \lambda^k(t_\mathrm{f} - t) O_i m^k(t_\mathrm{f} - t)) + g.\end{aligned}$$

This is end of the proof.

Appendix C. Proof of theorem 2

Lemma C.1. *If H_i is chosen as $H_i = \frac{1}{2}\Omega_i$, i.e. $O_i = 2T_{h_i} = T_{e_i}$, $i = 1, \ldots, N^2 - 1$, we have*

$$\begin{aligned}\mathrm{e}^{-O_0(t-t_0)} O_{2k-1}\,\mathrm{e}^{O_0(t-t_0)} &= O_{2k-1}\cos\omega(t - t_0) - O_{2k}\sin\omega(t - t_0), \\ \mathrm{e}^{-O_0(t-t_0)} O_{2k}\,\mathrm{e}^{O_0(t-t_0)} &= O_{2k-1}\sin\omega(t - t_0) + O_{2k}\cos\omega(t - t_0),\end{aligned}$$

where $k = 1, \ldots, r$ and e_i is the ith natural basis vector of $\mathcal{R}^{N^2-1}$.

Proof. For the special structure of O_0 in equation (4), it is known that

$$\begin{aligned}[H_0, H_{2k-1}] &= O_0\,\mathrm{e}_{2k-1}\cdot\vec{\Omega} = \omega_k\,\mathrm{e}_{2k}\cdot\vec{\Omega} = \omega_k H_{2k}, \\ [H_0, H_{2k}] &= O_0\,\mathrm{e}_{2k}\cdot\vec{\Omega} = -\omega_k\,\mathrm{e}_{2k-1}\cdot\vec{\Omega} = -\omega_k H_{2k-1},\end{aligned}$$

where $k = 1, \ldots, r$ and e_i is the ith natural basis vector of $\mathcal{R}^{N^2-1}$. Hence, for any $m \in \mathcal{R}^{N^2-1}$, we have

$$([O_0, O_{2k-1}]m)\cdot\vec{\Omega} = [[H_0, H_{2k-1}], m\cdot\vec{\Omega}] = [\omega_k H_{2k}, m\cdot\vec{\Omega}] = \omega_k O_{2k} m\cdot\vec{\Omega},$$

i.e. $[O_0, O_{2k-1}]m = \omega_k O_{2k} m$, which means $[O_0, O_{2k-1}] = \omega_k O_{2k}$. It is also easy to show that $[O_0, O_{2k}] = -\omega_k O_{2k-1}$. The lemma can then be proved by simple calculations from the equality $\mathrm{e}^A B\,\mathrm{e}^{-A} = \sum_{i=0}^{\infty}\frac{1}{i!}[A^{(i)}, B]$, where $[A^{(i)}, B] = [A^{(i-1)}, [A, B]]$ and $[A^{(0)}, B] = B$. □

Proof of theorem 2. It is sufficient to prove that, if ξ and η satisfy equation (10), $m_{\mathrm{ss}}(t) = \mathrm{e}^{O_0(t-t_0)}\xi$ and $\lambda_{\mathrm{ss}}(t) = \mathrm{e}^{O_0(t-t_0)}\eta$ will be the solution of (9). Substituting $m_{\mathrm{ss}}(t)$

and $\lambda_{ss}(t)$ into the first equation of (9), we have

$$\begin{aligned}\dot{m}_{ss}(t) &= -Dm_{ss}(t) + O_0 m_{ss}(t) + \sum_{i=1}^{N^2-1} u_i^{ss} O_i m_{ss} + g \\ &\Leftarrow O_0\,\mathrm{e}^{O_0(t-t_0)}\xi = -D\,\mathrm{e}^{O_0(t-t_0)}\xi + O_0\,\mathrm{e}^{O_0(t-t_0)}\xi + \sum_{i=1}^{N^2-1} u_i^{ss} O_i\,\mathrm{e}^{O_0(t-t_0)}\xi + g \\ &\Leftarrow -D\,\mathrm{e}^{O_0(t-t_0)}\xi + \sum_{i=1}^{N^2-1} u_i^{ss} O_i\,\mathrm{e}^{O_0(t-t_0)}\xi + g = 0.\end{aligned}$$

Since equation (1) satisfies (O) conditions, i.e. $[O_0, D] = 0, O_0 g = 0$, we have $[\mathrm{e}^{O_0(t-t_0)}, D] = 0$ and $\mathrm{e}^{O_0(t-t_0)} g = g$. Thus, the proof is reduced to showing that

$$\mathrm{e}^{O_0(t-t_0)}(-D\xi + g) + \sum_{i=1}^{N^2-1} u_i^{ss} O_i\,\mathrm{e}^{O_0(t-t_0)}\xi = 0.$$

From lemma C.1 and equations $[O_0, O_l] = 0, l = 2r+1, \ldots, N^2-1$, it is known that
$u_{2k-1}^{ss}(t) = \epsilon m_{ss}^T(t) O_{2k-1}\lambda_{ss}(t) = \epsilon\xi^T O_{2k-1}\eta\cos\omega_k(t-t_0) - \epsilon\xi^T O_{2k}\eta\sin\omega_k(t-t_0)$,
$u_{2k}^{ss}(t) = \epsilon m_{ss}^T(t) O_{2k}\lambda_{ss}(t) = \epsilon\xi^T O_{2k-1}\eta\sin\omega_k(t-t_0) + \epsilon\xi^T O_{2k}\eta\cos\omega_k(t-t_0)$,
$u_l^{ss}(t) = \epsilon m_{ss}^T(t) O_l\lambda_{ss}(t) = \epsilon\xi^T\,\mathrm{e}^{-O_0(t-t_0)} O_l\,\mathrm{e}^{O_0(t-t_0)}\eta = \epsilon\xi^T O_l\eta$,
where $k = 1, \ldots, r$ and $l = 2r+1, \ldots, N^2-1$. Thus, we have

$$\sum_{i=1}^{N^2-1} u_i^{ss} O_i m_{ss}(t) = \mathrm{e}^{O_0(t-t_0)} \sum_{i=1}^{N^2-1} \epsilon(\xi^T O_i\eta) O_i\xi.$$

In fact, it can be calculated from lemma 1 that

$$\begin{aligned}\sum_{i=1}^{2r} u_i^{ss} O_i m_{ss}(t) &= (O_1 m_{ss}, \ldots, O_{2r} m_{ss}, 0)\begin{pmatrix} u_1 \\ \vdots \\ u_{2r} \\ 0\end{pmatrix} \\ &= \mathrm{e}^{O_0(t-t_0)}(\mathrm{e}^{-O_0(t-t_0)} O_1\,\mathrm{e}^{O_0(t-t_0)}\xi, \ldots, \mathrm{e}^{-O_0(t-t_0)} O_{2r}\,\mathrm{e}^{O_0(t-t_0)}\xi, 0) \\ &\quad\times \mathrm{e}^{-O_0(t-t_0)}\begin{pmatrix} \xi^T O_1\eta \\ \vdots \\ \xi^T O_{2r}\eta \\ 0\end{pmatrix} \\ &= \mathrm{e}^{O_0(t-t_0)}(O_1\xi, \ldots, O_{2r}\xi, 0)\,\mathrm{e}^{O_0(t-t_0)}\,\mathrm{e}^{-O_0(t-t_0)}\begin{pmatrix} \xi^T O_1\eta \\ \vdots \\ \xi^T O_{2r}\eta \\ 0\end{pmatrix} \\ &= \mathrm{e}^{O_0(t-t_0)} \sum_{i=1}^{2r} \epsilon(\xi^T O_i\eta) O_i\xi,\end{aligned}$$

$$\sum_{l=2r+1}^{N^2-1} u_l^{ss} O_l m_{ss}(t) = \sum_{l=2r+1}^{N^2-1} \epsilon(\xi^T O_l\eta) O_l\,\mathrm{e}^{O_0(t-t_0)}\xi = \mathrm{e}^{O_0(t-t_0)} \sum_{l=2r+1}^{N^2-1} \epsilon(\xi^T O_l\eta) O_l\xi.$$

Therefore, $m_{ss}(t)$ and $\lambda_{ss}(t)$ are stationary solutions of (9), if and only if

$$\begin{cases} e^{O_0(t-t_0)}\left(-D\xi+\epsilon\sum_{i=1}^{N^2-1}(\xi^T O_i\eta)O_i\xi+g\right)=0,\\ e^{O_0(t-t_0)}\left(D\eta-\xi+\epsilon\sum_{i=1}^{N^2-1}(\xi^T O_i\eta)O_i\eta+m_0\right)=0,\end{cases}$$

which is obvious from equations (10). This is the end of the proof. □

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambrige: Cambridge University Press)

[2] Shor P W 1994 Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer *Proc. 35th Ann. Symp. on Foundations of Computer Science*

[3] Grover L K 1996 A fast quantum mechanics algorithm for database search *Proc. 28th ACM Symp. on Theory of Computation*

[4] Knill E *et al* 2000 *Nature* **404** 368

[5] Nakamura Y *et al* 1999 *Nature* **398** 786

[6] Yamamoto T *et al* 2003 *Nature* **425** 941

[7] Thorwart M and Hanggi P 2002 *Phys. Rev.* A **65** 012309

[8] Shor P W 1995 *Phys. Rev.* A **52** 2493
Steane A M 1995 *Proc. R. Soc. Lond* A **452** 2551
Knill E and Laflamme R 1997 *Phys. Rev.* A **55** 900

[9] Cirac J I *et al* 1996 *Science* **273** 1207

[10] Zurek W H *et al* 1996 *Phys. Rev. Lett.* **77** 4683

[11] Duan L M and Guo G C 1997 *Phys. Rev. Lett.* **79** 1953

[12] Lloyd S and Slotine J-J E 1998 *Phys. Rev. Lett.* **80** 4088

[13] Zanardi P and Rasetti M 1997 *Phys. Rev. Lett.* **79** 3306
Duan L M and Guo G C 1997 *Phys. Rev. Lett.* **79** 1953
Lidar D A, Chuang I L and Whaley K B 1998 *Phys. Rev. Lett.* **81** 2594

[14] Chuang I L and Yamamoto Y 1995 *Phys. Rev.* A **52** 3489

[15] Duan L M and Guo G C 1997 *Phys. Rev.* A **57** 2399

[16] Tombesi P and Vitali D 1995 *Phys. Rev.* A **51** 4913
Goetsch P, Tobesi P and Vitali D 1996 *Phys. Rev.* A **54** 4519
Vitali D, Tombesi P and Milburn G J 1997 *Phys. Rev. Lett.* **79** 2442
Vitali D, Tombesi P and Milburn G J 1998 *Phys. Rev.* A **57** 4930

[17] Fortunato M, Raimoud J M, Tombesi P and Vitali D 1999 *Phys. Rev.* A **60** 1687

[18] Viola L and Lloyd S 1998 *Phys. Rev.* A **58** 2733

[19] Viola L, Lloyd S and Knill E 1999 *Phys. Rev. Lett.* **83** 4888

[20] Viola L, Knill E and Lloyd S 1999 *Phys. Rev. Lett.* **82** 2417

[21] Qiao B, Ruda H E and Chang J F 2002 *J. Appl. Phys.* **91** 9368

[22] Rabitz H *et al* 2000 *Science* **288** 824

[23] Rabitz H, Hsieh M and Rosenthal C 2004 *Science* **303** 1998

[24] Khaneja N, Reiss T, Luy B and Glaser S J 2003 *J. Magn. Reson.* **162** 311

[25] Khaneja N, Luy B and Glaser S J 2003 Boundary of quantum evolution under decoherence *Proc. Natl Acad. Sci. USA* **100** 13162–6

[26] D'Alessandro D and Dobrovivitski V 2002 Control of a two-level open quantum system *Proc. 41st IEEE Conf. on Decision and Control* vol 40

[27] Saito S *et al* 2004 *Phys. Rev. Lett.* **93** 037001

[28] Alicki R and Lendi K 1985 *Quantum Dynamical Semigroup and Applications* (New York: Springer)

[29] Gardiner C W 1992 *Quantum Noise* (Berlin: Springer)

[30] Altafini C 2003 *J. Math. Phys.* **44** 2357

[31] Thorwart M, Paladino E and Grifoni M 2004 *Chem. Phys.* **296** 333

[32] Fedichkin L, Fedorov A and Privman V 2004 *Phys. Lett.* A **328** 87

[33] van Kampen N G 1995 *J. Stat. Phys.* **78** 299

[34] Privman V 2003 *J. Stat. Phys.* **110** 957
[35] Quang T, Woldeyohannes M, John S and Agarwal G S 1997 *Phys. Rev. Lett.* **79** 5238
John S and Quang T 1997 *Phys. Rev. Lett.* **78** 1888
[36] Thorwart M *et al* 2000 *J. Mod. Opt.* **47** 2905
[37] Ahn D, Lee J, Kim M S and Hwang S W 2002 *Phys. Rev.* A **66** 012302
[38] Huang G M, Tarn T J and Clark J 1983 *J. Math. Phys.* **24** 2608
Kunita H 1976 *Appl. Math. Optim.* **5** 89
[39] Athans M and Falb P 1996 *Optimal Control: An Introduction to the Theory and its Applications* (New York: McGraw-Hill)
[40] D'Alessandro D and Dahleh M 2001 *IEEE Trans. Autom. Control* **46** 866
[41] Wiseman H M 1994 *Phys. Rev.* A **49** 2133
Wiseman H M 1994 *Phys. Rev.* A **49** 5159 (erratum)
Wiseman H M 1994 *Phys. Rev.* A **50** 4428 (erratum)
[42] Wang J and Wiseman H M 2001 *Phys. Rev.* A **64** 063810
[43] Lan C H, Tarn T J, Chi Q S and Clark J W 2005 *J. Math. Phys.* **46** 052102
[44] Bayfield J E 1999 *Quantum Evolution: An Introduction to Time-Dependent Quantum Mechanics* (New York: Wiley)
[45] Thorwart M, Riemann P and Hanggi P 1999 *A Real-Time Path Integral Method for Driven Dissipative Quantum System* (Germany: Theoretische Physik I) pp 142–4
(http://www.lrz-muenchen.de/projekte/hlr-projects/1997-1999/cd/daten/pdf/b5101bm.pdf)